1. (a)

$I=\int_0^{\infty} e^{-x}dx=\lim_{k\to\infty}\int_0^k e^{-x}dx \Rightarrow I=\lim_{k\to\infty}[-e^{-x}]_0^k=\lim_{k\to\infty}[-e^{-k}+e^0]$

$\Rightarrow I=\lim_{k\to\infty}(1-e^{-k})=1-0=1 \quad [\because \lim_{k\to\infty} e^{-k}=e^{-\infty}=0]$ Thus,

$\lim_{k\to\infty}\int_0^k e^{-x}dx$ exists and is finite. Hence the given integral is convergent.

2. (a)

$I=\int_1^2 \sqrt{x-1}dx+\int_1^2 \frac{2}{\sqrt{x-1}}dx = \left[\frac{2}{3}(x-1)^{3/2}\right]_1^2+[4\sqrt{x-1}]_1^2=$

$14/3$ which is finite so convergent.

3. (c)

$I=\int_1^2 \frac{dx}{(x-1)(x-4)}=\frac{1}{3}\int_1^2\left(\frac{1}{x-4}-\frac{1}{x-1}\right)dx=$

$\frac{1}{3}[\log 2-\infty]=-\infty$

So the given integral is not convergent.

4. (b)

Let $I=\int_{-10}^{20}[\cot^{-1}x]\,dx,$

we know $\cot^{-1}x\in(0,\pi)\ \forall\ x\in R$

thus, $[\cot^{-1}x]=\begin{cases}3, & x\in(-\infty,\cot 3)\\ 2, & x\in(\cot 3,\cot 2)\\ 1 & x\in(\cot 2,\cot 1)\\ 0 & x\in(\cot 1,\infty)\end{cases}$

Hence, $I=\int_{-10}^{\cot 3}3\,dx+\int_{\cot 3}^{\cot 2}2\,dx+\int_{\cot 2}^{\cot 1}1\,dx+\int_{\cot 1}^{20}0\,dx$

$=30+\cot 1+\cot 2+\cot 3$

5. (a)

Let $I=\int_0^2 [x^2-x+1]\,dx$

$=\int_0^{\frac{1+\sqrt5}{2}}[x^2-x+1]\,dx+\int_{\frac{1+\sqrt5}{2}}^2[x^2-x+1]\,dx$

$=\int_0^{\frac{1+\sqrt5}{2}}1\,dx+\int_{\frac{1+\sqrt5}{2}}^2 2dx=\frac{7-\sqrt5}{2}$

6. (b)

$\frac{df(x)}{dx}=\cos\left(\sqrt{x}\right)^2\frac{d\sqrt{x}}{dx}-\cos\left(\frac{1}{x}\right)^2\cdot\frac{d}{dx}\left(\frac{1}{x}\right)$

$=\frac{1}{2\sqrt{x}}\cos x+\frac{\cos x^{-2}}{x^2}=\frac{x\sqrt{x}\cos x+2\cos(x^{-2})}{2x^2}$

7. (b)

$I=\int_0^{\pi} x\ f(\sin x)\,dx$

$=\int_0^{\pi}(\pi-x)\ f\sin(\pi-x)\,dx=\pi\int_0^{\pi}f(\sin x)\,dx-I$

$\Rightarrow I=\frac{\pi}{2}\int_0^{\pi}f(\sin x)\,dx.$

Again, $I=\frac{\pi}{2}\int_0^{\pi}f(\sin x)\,dx=2\ \frac{\pi}{2}\int_0^{\pi/2}f(\sin x)dx$

$=\pi\int_0^{\pi/2}f(\sin x)dx$

8. (b)

Let $I=\int_a^b x\ f(x)\,dx=\int_a^b (a+b-x)\,f(a+b-x)\,dx$

$=(a+b)\int_a^b f(a+b-x)dx-\int_a^b x\,f(a+b-x)\,dx$

$=(a+b)\int_a^b f(x)\,dx-\int_a^b x\,f(x)\,dx.$

Hence $I=\left(\frac{a+b}{2}\right)\int_a^b f(x)\,dx.$

9. (b)

Using the property $\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$, the given integral

$I=\int_{\pi/6}^{\pi/3}\frac{dx}{1+\tan^5 x}=\int_{\pi/6}^{\pi/3}\frac{dx}{1+\tan^5\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}$

$=\int_{\pi/6}^{\pi/3}\frac{dx}{1+\cot^5 x}.$

Hence $2I=\int_{\pi/6}^{\pi/3}dx\Rightarrow I=\frac{1}{2}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)=\frac{\pi}{12}$

10. (d)

$I=\int_{-1}^{1}\frac{x^2}{1+x^2}dx+0$ (because $\frac{\sin x}{1+x^2}$ is odd)

$I=2\int_0^1\frac{x^2+1-1}{x^2+1}\,dx=2\int_0^1 dx-2\int_0^1\frac{dx}{1+x^2}$

$=2-2\tan^{-1}x\Big|_0^1=2-\frac{\pi}{2}$

11. (c)

$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{r}{n+r}=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{\frac{r}{n}}{1+\frac{r}{n}}\right)$

$=\int_0^1\frac{x\,dx}{1+x}=\int_0^1 dx-\int_0^1\frac{dx}{1+x}=1-\ln 2$

12. (b)

Since the denominator is even and the numerator f(sin (-x)) = f (-sin x)= -f (sin x) is odd, (as f is odd function), the integral is equal to zero

13. (b)

Since $0 < 2e^{-x} \le 2 \ \forall\, x \in [0, \infty)$,

$[2e^{-x}] = 0$, for $x > \ln 2$.

Hence the given integral = $\int_0^{\ln 2} dx + 0 = \ln 2$

14. (a)

$$\int_0^{\pi/2} \operatorname{cosec}(x-\pi/3)\operatorname{cosec}(x-\pi/6)\,dx$$

$$= 2\int_0^{\pi/2} \frac{\sin\left[\left(x-\frac{\pi}{6}\right)-\left(x-\frac{\pi}{3}\right)\right]}{\sin\left(x-\frac{\pi}{6}\right).\sin\left(x-\frac{\pi}{3}\right)}dx$$

$$\Rightarrow 2\int_0^{\pi/2}\left[\cot\left(x-\frac{\pi}{3}\right)-\cot\left(x-\frac{\pi}{6}\right)\right]dx$$

$$\Rightarrow 2\left[\log\sin\left(x-\frac{\pi}{3}\right)-\log\sin\left(x-\frac{\pi}{6}\right)\right]_0^{\pi/2}$$

$$\Rightarrow 2\left[\log\left(\frac{\sin\left(x-\frac{\pi}{3}\right)}{\sin\left(x-\frac{\pi}{6}\right)}\right)\right]_0^{\pi/2} = 2\left[\log\left(\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)-\log\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right)\right]$$

$$\Rightarrow 2\left[-\log\sqrt{3}-\log\sqrt{3}\right]$$

$$\Rightarrow -4\log\sqrt{3} = -2\log 3.$$

15. (c)

We have,

$$I_{n+2}+I_n = \int_0^{\pi/4}\tan^{n+2}x\,dx + \int_0^{\pi/4}\tan^n x\,dx =$$

$$\int_0^{\pi/4}\tan^n x\left(1+\tan^2 x\right)dx = \int_0^{\pi/4}\tan^n x\sec^2 x\,dx =$$

$$\int_0^1 t^n\,dt, \text{where } t=\tan x$$

$$= \frac{1}{n+1}$$

16. (b)

Let $\ln x = t^2$

$x = e^{t^2} \Rightarrow dx = e^{t^2}.\,2t\,dt$

$$2\int_1^2 t^2.e^{t^2}.dt = \int_1^2 t(2te^{t^2})\,dt$$

$$= (t\,e^{t^2})_1^2 - \int_1^2 e^{t^2}dt = 2e^4 - e - \alpha$$

17. (c)

$$I_2 = \int_0^1 \frac{\tan^{-1}x}{x}dx,\ x=\tan\theta$$

$$\Rightarrow I_2 = \int_0^{\pi/4}\frac{2\theta}{\sin 2\theta}d\theta = \frac{1}{2}\int_0^{\pi/2}\frac{x}{\sin x}dx = \frac{1}{2}.I_1$$

$$\Rightarrow \frac{I_1}{I_2} = 2$$

18. (c)

Since $\frac{d}{dx}\left(\frac{1}{x\sin x+\cos x}\right) = \frac{-x\cos x}{(x\sin x+\cos x)^2}$

integration by parts,

$$I = \left[\frac{-x\sec x}{x\sin x+\cos x}+\tan x\right]_0^{\pi/4} = \frac{4-\pi}{4+\pi}$$

19. (c)

Note that $\int_0^{b-c} f(x+c)\,dx = -\int_b^c f(x)\,dx$

20. (c)

$$I_2 = \int_0^1 \frac{x^2\,dx}{e^{x^3}(2-x^3)}$$

{let $1 - x^3 = t \Rightarrow x^2dx = \frac{-dt}{3}$ }

$$= -\frac{1}{3}\int_1^0\frac{dt}{e^{1-t}(1+t)} = \frac{1}{3e}\int_0^1\frac{e^t}{1+t}dt$$

$$I_2 = \frac{1}{3e}I_1 \Rightarrow \frac{I_1}{I_2} = 3e$$