1. (d)

For $x \in [0, 2]$, $x^2 + 1 \in [1, 5]$,

we must break $[0, 2] = [0, 1] \cap [1, \sqrt{2}\,]$

$=[0, 1] \cup [1, \sqrt{2}\,] \cup [\sqrt{2}, \sqrt{3}\,] \cup [\sqrt{3}, 2]$.

Hence $\int_0^2 x^{[x^2+1]}\,dx$

$$= \int_0^1 x^{[x^2+1]}\,dx + \int_1^{\sqrt{2}} x^{[x^2+1]}\,dx + \int_{\sqrt{2}}^{\sqrt{3}} x^{[x^2+1]}\,dx + \int_{\sqrt{3}}^{2} x^{[x^2+1]}\,dx$$

$$= \int_0^1 x\,dx + \int_1^{\sqrt{2}} x^2 dx + \int_{\sqrt{2}}^{\sqrt{3}} x^3\,dx + \int_{\sqrt{3}}^{2} x^4 dx = \frac{1}{2} + \frac{1}{3}[2^{3/2} - 1] +$$

$$\frac{1}{4}[9 - 4] + \frac{1}{5}[32 - 3^{5/2}] = \frac{469}{60} + \frac{1}{3}2^{3/2} - \frac{1}{5}3^{5/2}.$$

2. (b)

$$\int_{-1}^{3}\{|x-2|+[x]\}\,dx = \int_{-1}^{0}\{|x-2|+[x]\}\,dx$$

$$+ \int_0^1\{|x-2|+[x]\}dx + \int_1^2\{|x-2|+[x]\}dx + \int_2^3\{|x-2|+[x]\}dx$$

$$= \int_{-1}^{0}(2-x-1)dx + \int_0^1(2-x+0)dx + \int_1^2(2-x+1)dx +$$

$$\int_2^3(x-2+2)dx$$

$$= \left. x - \frac{x^2}{2}\right|_{-1}^{0} + \left. 2x - \frac{x^2}{2}\right|_0^1 + \left. 3x - \frac{x^2}{2}\right|_1^2 + \left. \frac{x^2}{2}\right|_2^3$$

$$= -\left(-1-\frac{1}{2}\right) + \left(2-\frac{1}{2}\right) + (6 - 2) - \left(3-\frac{1}{2}\right) + \frac{9}{2} - 2 = 7.$$

3. (a)

$$I_m = \int_1^e (\log x)^m\,dx = x(\log x)^m\Big|_1^e$$

$$- m\int_1^e(\log x)^{m-1}dx$$

$$= e - m\left[x(\log x)^{m-1}\Big|_1^e - (m-1)\int_1^e(\log x)^{m-2}dx\right]$$

$= e - me + m(m - 1)I_{m-2} = (1 - m)e + m(m - 1)I_{m-2}$

So $\frac{I_m}{1-m} + m I_{m-2} = e$. Thus $K = 1 - m$ and $L = \frac{1}{m}$.

4. (d)

Integrating by parts, the given integral is equal to

$$x\tan^{-1}\sqrt{\sqrt{x}-1}\Big|_1^{16} - \int_1^{16}\frac{x}{\sqrt{x}}\,\frac{1}{4\sqrt{x}\sqrt{\sqrt{x}-1}}\,dx$$

$$= \frac{16}{3}\pi - \frac{1}{4}\int_1^{16}\frac{dx}{\sqrt{\sqrt{x}-1}}$$

$$= \frac{16}{3}\pi - \frac{1}{4}\int_0^{\sqrt{3}}\frac{4t(1+t^2)}{t}\,dt \quad (\sqrt{x} = 1 + t^2)$$

$$= \frac{16}{3}\pi - (\sqrt{3} + \sqrt{3}) = \frac{16\pi}{3} - 2\sqrt{3}.$$

5. (d)

$$\int_0^{\pi}[2\sin x]\,dx$$

$$= \int_0^{\pi/6}[2\sin x]dx + \int_{\pi/6}^{\pi/2}[2\sin x]dx$$

$$+ \int_{\pi/2}^{5\pi/6}[2\sin x]dx + \int_{5\pi/6}^{\pi}[2\sin x]dx$$

$$= \int_0^{\pi/6}0\,dx + \int_{\pi/6}^{\pi/2}1\,dx + \int_{\pi/2}^{5\pi/6}1\,dx + \int_{5\pi/6}^{\pi}0\,dx$$

$$= \frac{\pi}{2} - \frac{\pi}{6} + \frac{5\pi}{6} - \frac{\pi}{2} = \frac{2\pi}{3}.$$

6. (a)

$$\int_{-1/2}^{1/2}\left([x] + \log\frac{1+x}{1-x}\right)dx = \int_{-1/2}^{1/2}[x]dx$$

(since $\log\frac{1+x}{1-x}$ is an odd function)

$$= \int_{-1/2}^{0}[x]\,dx + \int_0^{1/2}[x]\,dx = \int_{-1/2}^{0}(-1)\,dx = -\frac{1}{2}.$$

7. (b)

$$\int_{-1/2}^{1/2}\sqrt{\left(\frac{x+1}{x-1} - \frac{x-1}{x+1}\right)^2}$$

$$= \int_{-1/2}^{1/2}\left|\frac{x+1}{x-1} - \frac{x-1}{x+1}\right|dx$$

$$= \int_{-1/2}^{1/2}\left|\frac{4x}{x^2-1}\right|dx = 2\int_0^{1/2}\left|\frac{4x}{x^2-1}\right|dx$$

($\because$ integrand is an even function)

$$= -2\int_0^{1/2}\left(\frac{4x}{x^2-1}\right)dx$$

$$\left(\because \frac{4x}{x^2-1} < 0 \text{ in the interval } \left(0, \frac{1}{2}\right)\right)$$

$$= -4[\log(1 - x^2)]_0^{1/2} = -4\left(\log\frac{3}{4}\right) = 4\log\left(\frac{4}{3}\right).$$

8. (b)

$$\int_0^{n^2}\left[\sqrt{x}\right]dx = \int_0^1\left[\sqrt{x}\right]dx + \int_1^4\left[\sqrt{x}\right]dx + \int_4^9\left[\sqrt{x}\right]dx + \ldots. \int_{(n-1)^2}^{n^2}\left[\sqrt{x}\right]$$

dx

$$\left[\because \left[\sqrt{x}\right] = 0, \text{if } 0 \le x < 1; 1, \text{if } 1 \le x < 4; \; 2, \text{if } 4 \le x < 9 \ldots, (n-1), \text{if } (n-1)^2 \le x < n^2\right]$$

$$= \int_0^1 0dx + \int_1^4 1dx + \int_4^9 2dx + \ldots + \int_{(n-1)^2}^{n^2}(n-1)\,dx$$

$= 1(4 - 1) + 2(9 - 4) + \ldots.. + (n - 1)[n^2 - (n - 1)^2]$

$= -(1^2 + 2^2 + 3^2 + \ldots. + n^2) + n^3$

$$= n^3 - \frac{n(n+1)(2n+1)}{6} = \frac{n(n-1)(4n+1)}{6}.$$

9. (a)

$$\int_0^{2[x]}(x-[x])dx = \int_0^{2[x].1}(x-[x])dx$$

$$= 2[x]\int_0^1(x-[x])dx$$

[$\because$ x - [x] is a periodic function of period 1]

$$= 2[x]\left(\left(\frac{x^2}{2}\right)_0^1 - \int_0^1[x]dx\right) = 2[x]\left(\frac{1}{2}-0\right) = [x].$$

10. (b)

$$I_1 = \int_{\sin^2 t}^{1+\cos^2 t} x\, f\{(2-x)\}\, dx$$

$$= I_1 = \int_{\sin^2 t}^{1+\cos^2 t}(2-x)\, f(x(2-x))\, dx = 2\,.\,I_2 - I_1$$

$$\Rightarrow 2I_1 = 2I_2 \;\Rightarrow \frac{I_1}{I_2} = 1.$$

11. (c)

$$\text{Let } I = \int_a^b \frac{|x|}{x}dx$$

Case I : When 0 < a < b

$$\text{then } I = \int_a^b \frac{x}{x}dx = \int_a^b 1\,.\,dx = x\Big|_a^b = b - a$$

$= |b| - |a|$ … (1)

Case II : When a < 0 < b

$$\text{then } I = -\int_a^0 1\,.\,dx + \int_0^b 1\,.\,dx$$

$= -(0 - a) + (b - 0)$

$= b + a$

$= |b| - |a|$ … (2)

Case III : When a < b < 0

$$\text{then } I = \int_0^b(-1)\,.\,dx = -(b - a) = -b + a$$

$= |b| - |a|$ … (3)

It is clear from (1), (2) and (3) we get

$$\int_0^b \frac{|x|}{x}dx = |b| - |a|.$$

12. (b)

$$I = \int_{4\pi-2}^{4\pi}\frac{\sin(t/2)}{4\pi+2-t}dt$$

$$= (4\pi - (4\pi - 2))\int_0^1 \frac{\sin\left(\frac{(4\pi-(4\pi-2))t+4\pi-2}{2}\right)}{4\pi+2-((4\pi-(4\pi-2)t+4\pi-2)}dt$$

$$= 2\int_0^1 \frac{\sin(t-1)}{(4-2t)}dt$$

$$= \int_0^1 \frac{\sin(t-1)}{(2-t)}dt$$

$$= \int_0^1 \frac{\sin(1-t-1)}{2-(1-t)} \quad \text{(By Prop. IV)}$$

$$= \int_0^1 \frac{\sin(-t)}{(1+t)}dt$$

$$= -\int_0^1 \frac{\sin t}{1+t}dt$$

$= -\alpha$. (given).

13. (b)

$$\text{Let } I = \int_0^{\frac{16\pi}{3}}|\sin x|$$

$$= \int_0^{5\pi+\frac{\pi}{3}}|\sin x|\ dx$$

$$= \int_0^{5\pi}|\sin x|dx + \int_{5\pi}^{5\pi+\frac{\pi}{3}}|\sin x|\ dx$$

$$= 5\int_0^{\pi}|\sin x|dx + \int_0^{\pi/3}|\sin x|\ dx$$

(By Prop. XIII & XV)

($\because$ | sin x| is periodic with period π)

$$= 5\int_0^{\pi}\sin x\ dx + \int_0^{\pi/3}\sin x\ dx$$

$$= 5(-\cos x)\Big|_0^{\pi} + (-\cos x)\Big|_0^{\pi/3}$$

$= 5(1+1) - \left(\frac{1}{2} - 1\right) = \frac{21}{2}$

14. (d)

Let $I = \int_0^1 \frac{\ln x}{(1+x)} dx$

Integrating by parts taking *ln* x as first function, we have

$= [\ln x . \ln(1+x)]_0^1 - \int_0^1 \frac{\ln(1+x)}{x} dx$

$I = 0 - \int_0^1 \frac{\ln(1+x)}{x} dx$

$= -\int_0^1 \left(\frac{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...}{x} \right) dx$

$= -\int_0^1 \left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + ...\right) dx$

$= -\left[x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + ... \right]_0^1$

$= -\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ...\right)$

$= -\frac{\pi^2}{12}$

15. (a)

Let $I = \int_0^1 \cot^{-1} (1 - x + x^2) dx$

$= \int_0^1 \cot^{-1} (1 - x(1-x)) dx$

$(\because 0 \le x < 1)$

$= \int_0^1 \tan^{-1} \left(\frac{1}{1 - x(1-x)} \right) dx$

$= \int_0^1 \tan^{-1} \left(\frac{x + (1-x)}{1 - x(1-x)} \right) dx$

$= \int_0^1 (\tan^{-1} x + \tan^{-1}(1-x)) dx$

$= \int_0^1 \tan^{-1} x \ dx + \int_0^1 \tan^{-1}(1-(1-x)) dx$

(From property)

$= 2\int_0^1 \tan^{-1} x \ dx$

Integrating by parts taking unity as the second function, we have

$I = 2\left[[\tan^{-1} x . x]_0^1 - \int_0^1 \frac{x}{1+x^2} dx \right]$

$= 2\left[\frac{\pi}{4} - \frac{1}{2}[\ln|1+x^2|]_0^1 \right]$

$= 2\left[\frac{\pi}{4} - \frac{1}{2}\ln 2 \right]$

Hence $I = \frac{\pi}{2} - \ln 2$.

16. (a)

Let $I = \int_{-1}^{1} [x[1 + \sin \pi x] + 1] dx,$

$= \int_{-1}^{0} [x[1 + \sin \pi x] + 1] dx + \int_{0}^{1} [x[1 + \sin \pi x] + 1] dx$

Now $[1 + \sin \pi x] = 0$ if $-1 < x < 0$

and $[1 + \sin \pi x] = 1$ if $0 < x < 1$

$\therefore I = \int_{-1}^{0} 1 . dx + \int_0^1 [x + 1] dx$

$= 1 + 1 \int_0^1 dx$

$= 1 + 1 = 2$

17. (a)

Let $f(x) = \int_0^x \cos t^2 dt$ and $g(x) = x$. Then $f(0) = g(0) = 0$.

$\therefore \lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)}$

$\Rightarrow \lim_{x\to 0} \frac{\int_0^x \cos t^2 dt}{x} = \lim_{x\to 0} \frac{\cos x^2 . 1 - \cos 0.0}{1}$

$= \lim_{x \to 0} \frac{\cos x^2}{1} = \cos 0 = 1.$

18. (b)

$\int_0^{102} [\tan^{-1} x] dx$

$= \int_0^{\tan 1} [\tan^{-1} x] dx + \int_{\tan 1}^{102} [\tan^{-1} x] dx$

$= 0 + 1. \int_{\tan 1}^{102} 1.dx$

$= 102 - \tan 1.$

19. (c)

$g(x+2) = \int_0^{x+2} f(t) \ dt$

$$\int_0^2 f(t)\,dt + \int_2^{x+2} f(t)\,dt = g(2) + \int_0^x f(t)\,dt$$

$\therefore g(x+2) = g(2) + g(x) \Rightarrow g(x)$ is periodic with period 2

Also, $g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt$

$$= \int_0^1 f(t)\,dt + \int_{-1}^0 f(t)\,dt$$

[putting t = u + 2]

$$= \int_{-1}^1 f(t)\,dt = 0 \quad [\because f(x) \text{ is odd}]$$

$\therefore g(2n) = 0 \quad [\because g(x)$ is periodic with period 2]

20. (b)

$$I_1 = \int_{\sin^2 t}^{1+\cos^2 t} x f(x(2-x))\,dx$$

$$= \int_{\sin^2 t}^{1+\cos^2 t} (2-x) f(x(2-x))\,dx = 2 \,.\, I_2 - I_1$$

$$2I_1 = 2I_2 \Rightarrow \frac{I_1}{I_2} = 1.$$

21. Ans. 3

Sol. use substitution $x + \sqrt{1+x^2} = t$ or $\left(\sqrt{1+x^2}\right)^2 = (t-x)^2$

22. Ans. 4

Sol. $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \sin^{2k}\left(\frac{r\pi}{2n}\right)$

$$= \int_0^1 \sin^{2k}\left(\frac{\pi x}{2}\right) dx \qquad \text{put } \frac{\pi x}{2} = \theta$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \sin^{2k}\theta\, d\theta$$

23. Ans. 7

Sol. $\int_0^{\pi/4} (\cos 2\theta)^{3/2} \cos\theta\, d\theta = \int_0^{\pi/4} (1 - 2\sin^2\theta)^{3/2} \cos\theta\, d\theta$

Put $\sin\theta = t$

$$I = \int_0^{1/\sqrt{2}} (1-2t^2)^{3/2} dt$$

again put $\sqrt{2}t = \sin z$

24. Ans. 3

Sol. $\int_0^{\pi/2} \frac{1+2\cos x}{(2+\cos x)^2} dx = \int_0^{\pi/2} \frac{1 + 2\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}{\left(2 + \dfrac{1-\tan^2 x/2}{1+\tan^2\frac{x}{2}}\right)^2} dx$

$$= \int_0^{\pi/2} \frac{\left(3-\tan^2\frac{x}{2}\right)\left(1+\tan^2\frac{x}{2}\right) dx}{\left(3+\tan^2\frac{x}{2}\right)^2}$$

Put $\tan\frac{x}{2} = t$

$$\Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$$

$$\int_0^1 \frac{2(3-t^2)\,dt}{(3+t^2)^2} = 2\int_0^1 \frac{\frac{3}{t^2} - 1}{\left(\frac{3}{t} + t\right)^2} dt$$

Put $t + \frac{3}{t} = z$

25. Ans. 1997

Sol. $I = \int_0^{\pi} x\, f(\cos^2 x + \tan^4 x)\,dx$

$$= \int_0^{\pi} (\pi - x)\, f(\cos^2 x + \tan^4 x)\,dx$$

$$\Rightarrow 2I = \int_0^{\pi} \pi\, f(\cos^2 x + \tan^4 x)\,dx$$

$$I = \frac{\pi}{2}\int_0^{\pi} f(\cos^2 x + \tan^4 x)\,dx$$

$$= \frac{\pi}{2} \times 2 \int_0^{\pi/2} f(\cos^2 x + \tan^4 x)\,dx$$