1. (b)

$(x^2/2)_0^a \le a+4 \Rightarrow a^2 \le 2a+8 \Rightarrow a^2-2a-8 \le 0$

$\Rightarrow (a-4)(a+2) \le 0 \Rightarrow -2 \le a \le 4$

2. (c)

Hence $f(x) = \dfrac{x^9 - 3x^5 + 7x^3 - 7x}{\cos^2 x} + \sec^2 x$

$\Rightarrow \displaystyle\int_{-\pi/a}^{\pi/a} \sec^2 x\,(x^9 - 3x^5 + 7x^3 - x) + \int_{-\pi/a}^{\pi/a} \sec^2 x\,dx$

$\Rightarrow \text{odd} + \displaystyle\int_{-\pi/a}^{\pi/a} \sec^2 x\,dx$

$\Rightarrow 2[\tan x]_0^{\pi/4} = 2\left[\tan\dfrac{\pi}{4} - \tan 0\right] = 2$

3. (d)

$\displaystyle\int_0^1 (a-x)^2 f(x)dx = a^2\int_0^1 f(x) - 2a\int_0^1 x.f(x) + \int_0^1 x^2.f'(x)$

$= a^2.1 - 2a.a + a^2 = 0$

4. (d)

$I = \displaystyle\int_{1/e}^{e} \frac{dt}{t(t^2+1)} = \int_{1/e}^{e}\left(\frac{1}{t} - \frac{t}{t^2+1}\right)dt$

$= [\log t]_{1/e}^{e} - \dfrac{1}{2}\left[\log(t^2+1)\right]_{1/e}^{e} = 1$

5. (b)

$I = \displaystyle\int_{-1}^{3}\left[\tan^{-1}\frac{x}{x^2+1} + \cot^{-1}\frac{x}{x^2+1}\right]dx$

$\Rightarrow \quad I = \displaystyle\int_{-1}^{3}\frac{\pi}{2}dx \Rightarrow \quad I = \frac{\pi}{2}[x]_{-1}^{3} = \frac{\pi}{2}[3+1] = 2\pi$.

6. (c)

$I = \displaystyle\int_3^6 \frac{1}{x+1}dx = [\log(x+1)]_3^6$,

$I = \displaystyle\int_3^6 \frac{1}{t+1}dt = [\log(t+1)]_3^6$

(2) $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ *i.e.*, by the interchange in the limits of definite integral, the sign of the integral is changed.

7. (b)

Let, $f_1(x) = \cos^3 x = -f(\pi - x)$ and

$f_2(x) = \cos^3(2n+1)x = -f(\pi - x)$

$I = 0$.

8. (a)

$I = \displaystyle\int_0^{2a}\frac{f(x)}{f(x)+f(2a-x)}dx = \int_0^{2a}\frac{f(2a-x)}{f(2a-x)+f(x)}dx$

$2I = \displaystyle\int_0^{2a}\frac{f(x)+f(2a-x)}{f(x)+f(2a-x)}dx = \int_0^{2a}dx = [x]_0^{2a} = 2a \quad \therefore I = a$.

9. (a)

We know, $\displaystyle\int_0^{\pi/2}\frac{\tan^n x dx}{1+\tan^n x} = \frac{\pi}{4}$ for any value of n

$\therefore \quad I = \pi/4$.

(5) $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_0^a f(x) + f(-x)\,dx$.

In special case :

$\displaystyle\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is even function or } f(-x) = f(x) \\ 0 & ,\text{if } f(x) \text{ is odd is odd function or } f(-x) = -f(x) \end{cases}$ This property is generally used when integrand is either even or odd function of x.

10. (a)

$\log\left(\dfrac{1+x}{1-x}\right)$ is an odd function of x as $f(-x) = -f(x)$

$I = \displaystyle\int_{-1/2}^{1/2}[x]dx + 0 \Rightarrow I = \int_{-1/2}^{0}[x]dx + \int_0^{1/2}[x]dx$

$\Rightarrow I = \displaystyle\int_{-1/2}^{0} -1dx + 0 \Rightarrow -[x]_{-1/2}^{0} = \frac{-1}{2}$.

11. (a)

$f(x) = \log[x + \sqrt{x^2+1}]$ is a odd function *i.e.* $f(-x) = -f(x)$

$\Rightarrow f(x) + f(-x) = 0 \Rightarrow I = 0$.

12. (d)

$I = \displaystyle\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx$

Put $x = 2 + 3 - t \Rightarrow dx = -dt$

$\therefore \quad I = \displaystyle\int_3^2 \frac{\sqrt{5-t}}{\sqrt{5-t}+\sqrt{t}}(-dt) = \int_2^3 \frac{\sqrt{5-x}}{\sqrt{5-x}+\sqrt{x}}dx$ and

$2I = \displaystyle\int_2^3 \frac{\sqrt{x}+\sqrt{5-x}}{\sqrt{5-x}+\sqrt{x}}dx = \int_2^3 1\,dx$

$\Rightarrow \quad 2I = [x]_2^3 = 1 \Rightarrow I = 1/2$

13. (b)

$I = \displaystyle\int_a^b x\,f(x)dx$ and $I = \displaystyle\int_a^b (a+b-x)f(a+b-x)dx$

$\Rightarrow I = \displaystyle\int_a^b (a+b-x)f(x)dx \Rightarrow I = \int_a^b (a+b)f(x)dx - \int_a^b x\,f(x)dx$

$\Rightarrow 2I = \left[\displaystyle\int_a^b f(x)dx\right](a+b) \Rightarrow I = \frac{a+b}{2}\int_a^b f(x)dx$

(8) $\displaystyle\int_0^a x\,f(x)dx = \frac{1}{2}a\int_0^a f(x)dx$ if $f(a-x) = f(x)$

14. (b)

Given, $\displaystyle\int_0^{\pi} x\,f(\sin x)dx = k\int_0^{\pi} f(\sin x)dx$

$\Rightarrow \displaystyle\int_0^{\pi}(\pi - x)f(\sin(\pi - x))dx = k\int_0^{\pi} f(\sin(\pi - x))dx$

$\Rightarrow \pi\displaystyle\int_0^{\pi} f(\sin x)dx - \int_0^{\pi} x\,f(\sin x)dx = k\int_0^{\pi} f(\sin x)dx \Rightarrow$

$\pi\displaystyle\int_0^{\pi} f(\sin x)dx - 2k\int_0^{\pi} f(\sin x)dx = 0 \Rightarrow (\pi - 2k)\int_0^{\pi} f(\sin x)dx = 0$

$\therefore \pi - 2k = 0 \Rightarrow k = \pi/2$

15. (a)

$I = \displaystyle\int_0^{2\pi}\frac{x\sin^{2n} x dx}{\sin^{2n} x + \cos^{2n} x}$ and

$$I=\int_0^{2\pi}\frac{(2\pi-x)\sin^{2n}(2\pi-x)dx}{\sin^{2n}(2\pi-x)+\cos^{2n}(2\pi-x)}$$

$$\left[\because \int_0^a f(x)=\int_0^a f(a-x)\right]$$

$$\therefore\ 2I=2\pi\int_0^{2\pi}\frac{\sin^{2n}\pi}{\sin^{2n}x+\cos^{2n}x}dx$$

$$\Rightarrow\ I=\pi\int_0^{2\pi}\frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx \text{ using } \int_0^{nT}f(x)=n\int_0^T f(x)dx$$

$$\therefore\ I=4\pi\int_0^{\pi/2}\frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx \Rightarrow I=4\pi(\pi/4)=\pi^2.$$

16. (c)

Consider the function $g(a)=\int_a^{a+T}f(x)dx$ =

$$\int_a^0 f(x)dx+\int_0^T f(x)dx+\int_T^{a+T}f(x)dx$$

Putting $x-T=y$ in last integral, we get

$$\int_T^{a+T}f(x)dx=\int_0^a f(y+T)dy=\int_0^a f(y)dy$$

$$\Rightarrow\ g(a)=\int_a^0 f(x)dx+\int_0^T f(x)dx+\int_0^a f(x)dx = \int_0^T f(x)dx$$

Hence $g(a)$ is independent of a.

17. (b)

$$\sum\lim_{n\to\infty}\frac{1}{r+\sqrt{rn}}=\sum\lim_{n\to\infty}\frac{1}{n\left[\frac{r}{n}+\sqrt{\frac{r}{n}}\right]}$$

$$\therefore\ \lim_{n\to\infty}S_n=\int_0^1\frac{1}{\sqrt{x}(1+\sqrt{x})}dx$$

$$=2[\log(1+\sqrt{x})]_0^1=2\log 2$$

18. (b)

Let $A=\lim_{n\to\infty}\frac{(n!)^{1/n}}{n}$

$$\Rightarrow\ \log A=\lim_{n\to\infty}\log\left(\frac{1.2.3......n}{n^n}\right)^{1/n}$$

$$\Rightarrow\ \log A=\lim_{n\to\infty}\log\left(\frac{1}{n}.\frac{2}{n}.\frac{3}{n}....\frac{n}{n}\right)^{1/n}$$

$$\Rightarrow\ \log A=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left[\log\left(\frac{r}{n}\right)\right]$$

$$\Rightarrow\ \log A=\int_0^1\log x dx=[x\log x-x]_0^1 \Rightarrow \log A=-1 \Rightarrow$$

$$A=e^{-1}$$

19. (c)

$$I=\frac{(4-1).(4-3).(6-1).(6-3).(6-5)}{(4+6)(4+6-2)(4+6-4)(4+6-6)(4+6-8)}\cdot\frac{\pi}{2}=\frac{3.1.5.3.1}{10.8.6.4.2.}\cdot\frac{\pi}{2}=\frac{3\pi}{512}$$

(1) $\int_0^\infty e^{-ax}\sin bx dx=\frac{b}{a^2+b^2}$

(2) $\int_0^\infty e^{-ax}\cos bx dx=\frac{a}{a^2+b^2}$

(3) $\int_0^\infty e^{-ax}x^n dx=\frac{n!}{a^n+1}$

20. (c)

Put, $\lambda x=t,\ \lambda dx=dt,$ we get,

$$\int_0^\infty e^{-\lambda x}x^{n-1}dx=\frac{1}{\lambda^n}\int_0^\infty e^{-t}t^{n-1}dt=\frac{1}{\lambda^n}\int_0^\infty e^{-x}x^{n-1}dx=\frac{I_n}{\lambda^n}$$